

Compactness for holomorphic curves with switching Lagrangian boundary conditions

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12 March 2009

Abstract

We prove a compactness result for holomorphic curves with boundary on an immersed Lagrangian submanifold with clean self-intersection. As a consequence, we show that the number of intersections of such holomorphic curves with the self-intersection locus is uniformly bounded in terms of the Hofer energy.

1 Introduction

In this paper we prove a compactness result for holomorphic curves with boundary on an immersed Lagrangian submanifold with clean self-intersection along a compact submanifold K . As a consequence, we show that the number of intersections of such holomorphic curves with K is uniformly bounded in terms of the Hofer energy. This finiteness result is an essential ingredient in the proof in [6] of the isomorphism of degree 0 Legendrian contact homology of the unit conormal bundle of a knot $K \subset \mathbb{R}^3$ with the cord algebra defined in [16].

Consider a symplectic manifold (X, ω) and an immersed Lagrangian submanifold $L \subset X$ with clean self-intersection along a compact submanifold K . Let J be an ω -compatible almost complex structure on X . We assume that near K the structure J is integrable and L is real analytic. Let (S, j) be a connected Riemann surface with boundary ∂S . A *holomorphic curve* $f : (S, \partial S, j) \rightarrow (X, L, J)$ is a continuous map $f : S \rightarrow X$ which maps ∂S to L and is (j, J) -holomorphic in the interior. We allow (X, L, ω, J) to be noncompact with cylindrical ends as in [9], and S to have punctures in the interior as well as on the boundary (see Section 3 for the precise setup). However, we do not treat intersections of $f|_{\partial S}$ with K – which we call *switches* – as boundary punctures. In particular, we do not impose any constraints on the number and types of switches.

Our first result states that the compactness result in symplectic field theory ([3, 7]) carries over to this setting. See Section 4 for the precise statement.

Theorem 1.1. *Under suitable hypotheses on (X, L, ω, J) each sequence of holomorphic curves $f_n : (S_n, \partial S_n, j_n) \rightarrow (X, L, J)$ of fixed signature*

and uniformly bounded energy has a subsequence converging in the sense of [3] to a stable holomorphic curve.

As a consequence, we obtain the following finiteness result for the number of switches.

Theorem 1.2. *In the situation of Theorem 1.1, suppose in addition that $(X, L, \omega = d\lambda)$ is exact with convex end. Then for each $s \in \mathbb{N}$ and $C > 0$ there exists a constant $\kappa(s, C) \in \mathbb{N}$ such that every holomorphic disk $f : (\dot{D}, \partial\dot{D}, j) \rightarrow (X, L, J)$ with at most s boundary punctures and energy $\leq C$ has at most $\kappa(s, C)$ switches.*

The case $s = 1$ of this finiteness result is an essential ingredient in the proof in [6] of the isomorphism of degree 0 Legendrian contact homology of the unit conormal bundle of a knot $K \subset \mathbb{R}^3$ with the cord algebra defined in [16]. This isomorphism is constructed by counting 1-punctured holomorphic disks in $T^*\mathbb{R}^3$ with boundary on the immersed Lagrangian submanifold $L = NK \cup \mathbb{R}^3$, where the conormal bundle $NK \subset T^*\mathbb{R}^3$ of K and the zero section \mathbb{R}^3 intersect cleanly along the knot K .

Holomorphic disks with boundary on cleanly intersecting Lagrangian submanifolds are also studied in [2].

To put Theorem 1.2 into context, recall that in general energy bounds are not enough to provide bounds on the topology of holomorphic curves. Indeed, double branched covers of $\mathbb{C}P^1$ exist for all genera, and by choosing the branch points to lie on the equator and cutting the domain along preimages of suitable segments connecting adjacent branch points, one obtains existence of holomorphic curves of genus zero and arbitrarily many boundary components, but of fixed energy.

Often, one can use index arguments to show that such phenomena disappear after suitable perturbation. Indeed, for the Fredholm theory of holomorphic curves $f : (S, \partial S) \rightarrow (X, L)$ as above, it is convenient to puncture the source S at points in ∂S that map to the clean intersection and call such punctures *Lagrangian intersection punctures*. It turns out that to each such puncture one can associate a winding number $w \in \frac{1}{2}\mathbb{N}$, and that the contribution of a Lagrangian intersection puncture in a clean intersection of codimension d to the Fredholm index is $1 - wd$ (see the appendix for more details). Consequently, this contribution is negative provided $d \geq 3$, and equal to 0 when $d = 2$ and $w = \frac{1}{2}$. It follows that for clean intersections of codimension at least three one can control the number of switches using transversality arguments. However, for codimension two – which is the most interesting case from the point of view of smooth embedding theory [6] – no such argument is available. Still, the result of this paper provides a bound on the number of switches which is independent of codimension.

Similar remarks apply to the number of boundary circles r and the genus g of S : If $\dim(L) = n$ satisfies $n > 3$ then the number of boundary circles and the genus can be bounded using transversality arguments; this again follows from the dimension formula for the corresponding moduli spaces (see the appendix). However, if $n = 3$ the dimension is independent of g and r and no such argument is available. Indeed, the contribution to the Gromov-Witten invariant of a Calabi-Yau 3-fold of multiple covers of

degree d and genus g of a fixed rational curve has been computed in [10]; it is nontrivial for any fixed $d \geq 2$ and arbitrarily high genus g , so there is no bound of the genus in terms of the degree. It would be interesting to have similar formulae for multiple covers of genus zero and many boundary components of a fixed (punctured) disk.

Our method of proof uses the integrability of J near K in an essential way. It would be interesting to understand to what extent the conclusion of Theorem 1.1 remains true for more general almost complex structures.

2 Local theory

Let (S, j) be a connected Riemann surface with boundary ∂S , possibly noncompact. We will consider functions $f : S \rightarrow \mathbb{C}$ satisfying the following conditions:

- (F1) f is continuous on S ;
- (F2) f is holomorphic on $\text{int } S := S \setminus \partial S$;
- (F3) f maps ∂S to $\mathbb{R} \cup i\mathbb{R}$.

We start with some elementary observations.

Lemma 2.1. *A function f satisfying (F1-3) is holomorphic on $\text{int } S \cup (\partial S \setminus f^{-1}(0))$.*

Proof. The function $g := f^2 : S \rightarrow \mathbb{C}$ is continuous on S , holomorphic on $\text{int } S$ and maps ∂S to \mathbb{R} , so by the Schwarz reflection principle it is holomorphic on all of S . Since the square root has holomorphic branches outside zero, the result for f follows. \square

Lemma 2.2. *If $f : S \rightarrow \mathbb{C}$ satisfying (F1-3) is not identically zero, then it has only finitely many zeroes in any compact subdomain $S' \subset S$.*

Proof. If not, then $g = f^2$ is a holomorphic function for which $g^{-1}(0)$ has a limit point, forcing it to vanish identically. \square

For $f : S \rightarrow \mathbb{C}$ satisfying (F1-3) and not identically zero, let γ be a path in S which does not meet any zero of f . Define the *winding number* of f along γ by

$$w(f, \gamma) := \frac{1}{2\pi} \int_{\gamma} f^* d\theta,$$

where $d\theta$ denotes the angular form on $\mathbb{C} \setminus \{0\}$.

Lemma 2.3. *Suppose $f : S \rightarrow \mathbb{C}$ satisfies (F1-3) and is not identically zero. Let $S' \subset S$ be a compact subset with piecewise smooth boundary $\partial S' = (S' \cap \partial S) \cup \Gamma$, where Γ is a union of disjoint arcs in S not meeting any zero of f . Then*

$$w(f, \Gamma) \geq \#(f^{-1}(0) \cap \text{int } S') + \frac{1}{4} \#(f^{-1}(0) \cap S' \cap \partial S).$$

Proof. Around each zero $p \in \text{int } S'$ pick a small disk $D_p \subset \text{int } S'$ containing no other zero. Then $w(f, \partial D_p) = k \in \mathbb{N}$, where $(z - p)^k$ is the first nonvanishing term in the power series expansion of f at p . Around each zero $q \in S' \cap \partial S$ pick a small half-disk $D_q^+ \subset S' \setminus (\partial S' \cap \text{int } S)$ containing no other zero and set $\partial^+ D_q^+ := \partial D_q \setminus \partial S$. Then $w(f, \partial^+ D_q^+) = k/4 \in \mathbb{N}/4$, where $(z - q)^k$ is the first nonvanishing term in the power series expansion of the holomorphic function $g = f^2$ at q . Now let S'' be the region obtained by removing from S' all disks resp. half-disks around zeroes of f . Since $d\theta$ is closed and the angle $f^*\theta$ is constant along parts of ∂S containing no zeroes, Stokes' theorem yields

$$0 = w(f, \partial S'') = w(f, \Gamma) - \sum_p w(f, \partial D_p) - \sum_q (f, \partial^+ D_q^+),$$

from which the lemma follows. \square

Lemma 2.4. *Let $f_n : S \rightarrow \mathbb{C}$ be a sequence of functions satisfying (F1-3), and assume that there is a constant $C > 0$ such that for all $n \geq 1$ and all $z \in S$ we have*

$$|f_n(z)| \leq C \quad (1)$$

Then there exists a subsequence $f_{n'}$ of the f_n , and a function $f : S \rightarrow \mathbb{C}$ satisfying (F1-3) such that

- (i) $f_{n'} \rightarrow f$ in C_{loc}^0 on S , and
- (ii) $f_{n'} \rightarrow f$ in C_{loc}^∞ on $\text{int } S \cup (\partial S \setminus f^{-1}(0))$.

Proof. Consider the associated sequence of holomorphic functions $g_n := f_n^2 : S \rightarrow \mathbb{C}$. The assumptions imply that for all $z \in S$ we have

$$|g_n(z)| \leq C^2.$$

Hence by Montel's theorem, after passing to a subsequence, the g_n converge in $C_{\text{loc}}^\infty(S)$ to a limit function $g : S \rightarrow \mathbb{C}$ which is holomorphic and maps ∂S to \mathbb{R} . By the same argument, after passing to a further subsequence, the f_n converge in $C_{\text{loc}}^\infty(\text{int } S)$ to a holomorphic function $f : \text{int } S \rightarrow \mathbb{C}$ satisfying $f^2 = g|_{\text{int } S}$.

At points $z \in \partial S$ with $g(z) \neq 0$ we extend f by taking the branch of \sqrt{g} that agrees with f at interior points near z , and at points $z \in \partial S$ with $g(z) = 0$ we set $f(z) := 0$. The resulting function $f : S \rightarrow \mathbb{C}$ satisfies (F1-3). In particular, Lemma 2.1 applies to show that f is holomorphic on $\text{int } S \cup (\partial S \setminus f^{-1}(0))$.

C_{loc}^0 -convergence of the f_n to f follows from the C_{loc}^0 -convergence of the g_n to g and continuity of the square root. It remains to show C_{loc}^∞ -convergence $f_n \rightarrow f$ on compact subsets of $\text{int } S \cup (\partial S \setminus f^{-1}(0))$. If $f \equiv 0$ this holds trivially, so suppose the f does not vanish identically. Fix a compact subset $S' \subset \text{int } S \cup (\partial S \setminus f^{-1}(0))$. By Lemma 2.2, f has only finitely many zeroes in S' . Pick a compact subset $S_0 \subset S' \cap \text{int } S$ containing all the zeroes and set $S_1 := S' \setminus \text{int } S_0$. On S_0 the C^∞ -convergence $f_n \rightarrow f$ was shown above, and on S_1 it follows from the C_{loc}^∞ -convergence $g_n \rightarrow g$ and smoothness of the square root away from zero. \square

The following statement is a variant of a result known as Vitali's theorem.

Lemma 2.5. *Let $f_n : S \rightarrow \mathbb{C}$ be a sequence of functions satisfying the assumptions of Lemma 2.4, and suppose there exists a compact subset $A \subset S$ such that each f_n has at least n zeroes in A . Then the limiting function f vanishes identically.*

Proof. Pick a compact subset $S' \subset S$ with piecewise smooth boundary $\partial S' = (S' \cap \partial S) \cup \Gamma$ such that $A \subset S' \setminus \Gamma$. If f has infinitely many zeroes in S' , then by Lemma 2.2 it vanishes. Otherwise, after passing to a subsequence, we may assume that f as well as each f_n has only finitely many zeroes in S' . After slightly shrinking S' we may assume that Γ avoids the countably many zeroes of f and the f_n . Since $A \subset S' \setminus \Gamma$ and f_n has at least n zeroes in A , Lemma 2.3 yields $w(f, \Gamma) \geq n/4$. On the other hand, since $f_n|_\Gamma$ converges smoothly to $f|_\Gamma$, we have

$$w(f_n, \Gamma) \xrightarrow{n \rightarrow \infty} w(f, \Gamma) < \infty,$$

contradicting the previous estimate. \square

3 Global theory

3.1 Setup

For the global theory we consider the following setup.

(X) (X, J) is an almost complex manifold with cylindrical end $\mathbb{R}_+ \times M$ adjusted to (ω, λ) in the sense of [3].

This means that $X = \bar{X} \cup (\mathbb{R}_+ \times M)$ with $\partial \bar{X} = M$, (ω, λ) is a stable Hamiltonian structure on M , ω extends to a symplectic form on \bar{X} , and J is compatible with ω on \bar{X} and with (ω, λ) on $\mathbb{R}_+ \times M$. We allow X to be noncompact but impose the following condition.

(Y) There exists a compact subset $\bar{Y} \subset \bar{X}$ such that every J -holomorphic map $f : S \rightarrow X$ from a compact Riemann surface with boundary satisfying $f(\partial S) \subset Y := \bar{Y} \cup (\mathbb{R}_+ \times (\bar{Y} \cap M))$ is entirely contained in Y .

Note that condition (Y) is trivially satisfied (taking $\bar{Y} = \bar{X}$) if \bar{X} is compact. Our assumption on the Lagrangian is the following.

(L) $L \subset Y \subset X$ is a properly immersed Lagrangian submanifold with $L \cap (\mathbb{R}_+ \times M) = \mathbb{R}_+ \times \Lambda$ for a compact submanifold $\Lambda \subset M$ satisfying $\lambda|_\Lambda = \omega|_\Lambda = 0$, and such that L has clean self-intersection along a compact connected submanifold $K \subset \text{int } \bar{Y}$.

Here *clean self-intersection* means that at each point $x \in K$ exactly two branches L_0, L_1 of L meet and $T_x K = T_x L_0 \cap T_x L_1$. More precisely, L is the image of a Lagrangian immersion $f : \tilde{L} \rightarrow X$ with clean self-intersection along the submanifold $\tilde{K} = f^{-1}(K)$. Then $f|_{\tilde{K}} : \tilde{K} \rightarrow K$ is a 2-1 covering and the two branches of L near $x \in K$ are the images under f of neighbourhoods of the preimages x_0, x_1 of x . Note that L may be 2-sheeted near K , i.e. the union of two embedded submanifolds intersecting

in K (if \tilde{K} is disconnected), or 1-sheeted (if \tilde{K} is connected). We impose the following condition on the almost complex structure near K .

- (K) There exists a neighbourhood \mathcal{U} of K on which J is integrable, a holomorphic embedding $\iota : K^{\mathbb{C}} \hookrightarrow X$ of a complexification of K , and a holomorphic projection $\tau : \mathcal{U} \rightarrow K^{\mathbb{C}}$ on a neighbourhood of K such that $\tau \circ \iota = \mathbb{1}$. Moreover, near every point $x \in K$ there exist holomorphic coordinates in $\mathbb{C}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k} \oplus i\mathbb{R}^k \oplus i\mathbb{R}^{n-k}$ sending x to 0, L_0 to \mathbb{R}^n and L_1 to $\mathbb{R}^k \oplus i\mathbb{R}^{n-k}$.

In particular, this implies that L is real analytic near K with J -orthogonal self-intersection along K , i.e. for every $x \in K$ the intersection $T_x L_0 \cap J(T_x L_1)$ is $(n - k)$ -dimensional. However, condition (K) is more restrictive than this. Indeed, not every pair of real analytic curves in \mathbb{C} intersecting orthogonally at the origin can be mapped to the coordinate axes by a local biholomorphism (e.g. if one curve is the y -axis, then the existence of such a biholomorphism imposes infinitely many constraints on the Taylor coefficients of the other curve as a graph over the x -axis).

Finally, we assume that the Reeb flow on M satisfies the following nondegeneracy condition:

- (R) No closed Reeb orbit meets Λ , and all closed Reeb orbits and Reeb chords are non-degenerate.

Here a *Reeb chord* is a Reeb orbit $\gamma : [0, T] \rightarrow M$ with $\gamma(0), \gamma(T) \in \Lambda$. If there are no closed Reeb orbits (e.g. for conormal lifts of $K \subset \mathbb{R}^n$ with the flat metric) these conditions can be arranged by a perturbation of Λ . In the contact case $\omega = d\lambda$ these conditions can be arranged by a perturbation of λ .

Our main case of interest is described in the following example.

Example 3.1 (cotangent bundle). Here the symplectic manifold $X = T^*Q$ is the cotangent bundle of a Riemannian manifold Q with the Liouville 1-form $\lambda = p dq$ and symplectic form $\omega = d\lambda$. $M = S^*Q$ is the unit cotangent bundle and J is the almost complex structure on T^*Q induced by the Riemannian metric, deformed outside S^*Q to make it cylindrical. $K \subset Q$ is a compact submanifold and $L = Q \cup NK$, where Q is the zero section and NK the conormal bundle, and $\Lambda = NK \cap S^*Q$. Then Q and NK intersect cleanly along K . We assume that $Q = \bar{Q} \cup (\mathbb{R}_+ \times \partial\bar{Q})$ with compact \bar{Q} ; then condition (Y) can be arranged (with $Y = T^*Q$) by making all level sets $\{r\} \times \partial\bar{Q}$ in the cylindrical end $\mathbb{R}_+ \times \partial\bar{Q}$ totally geodesic (then their preimages in T^*Q are Levi-flat and holomorphic curves cannot touch them from inside). Condition (R) holds for a generic metric; condition (K) can be arranged by Proposition 3.2 below, or by Remark 3.2 if K admits a flat metric and has trivial normal bundle (e.g. for a 1-knot in \mathbb{R}^3).

3.2 Structure near K

In this subsection we show that condition (K) can always be arranged by a deformation of the compatible almost complex structure near K , provided that L is 2-sheeted near K .

Proposition 3.2. *Let L_0, L_1 be Lagrangian submanifolds of a symplectic $2n$ -manifold (X, ω) intersecting cleanly along a closed submanifold K of dimension k . Then there exists an ω -compatible integrable complex structure J on a neighbourhood \mathcal{U} of K such that condition (K) holds.*

The proof of this proposition is based on three lemmata. The first one provides a symplectic normal form for L_0, L_1 near K .

Lemma 3.3. *Let L_0, L_1 be Lagrangian submanifolds of a symplectic $2n$ -manifold (X, ω) intersecting cleanly along a closed submanifold K of dimension k . Then there exists a symplectomorphism from a neighbourhood \mathcal{U} of K onto a neighbourhood of K in $(T^*L_0, \omega_{\text{st}})$ mapping L_0 to the zero section and L_1 to the conormal bundle NK .*

Proof. Consider the cotangent bundle $\pi : T^*L_0 \rightarrow L_0$ with its standard symplectic form ω_{st} . By the Lagrangian neighbourhood theorem, there exists a symplectomorphism from a neighbourhood (\mathcal{U}, ω) of K onto a neighbourhood of K in $(T^*L_0, \omega_{\text{st}})$ mapping L_1 to the conormal bundle NK . A short computation shows that the image L'_0 of L_0 under this symplectomorphism is tangent to the zero section L_0 along K . Thus after shrinking the neighbourhood we may assume that L'_0 is the graph of a closed 1-form λ . Since λ vanishes along K it equals dh for a function h whose differential vanishes along K . The Hamiltonian flow of $h \circ \pi : T^*L_0 \rightarrow \mathbb{R}$ is given by $\phi_t(q, p) = (q, p + t d_q h)$. So the time-(-1)-map ϕ_{-1} preserves NK and maps L'_0 to the zero section. \square

Next we construct a holomorphic model for L_0, L_1 near K for which condition (K) holds. Consider a complex vector bundle $E \rightarrow M$. A *holomorphic structure* on E is given by the structure of complex manifolds on E and M together with holomorphic local trivializations. By a *Kähler structure* on a holomorphic vector bundle E we mean a fibrewise linear Kähler form ω_E on E .

Lemma 3.4. *Let $F \rightarrow K$ be a real vector bundle over a compact manifold K and $E \rightarrow TK$ the pullback of the complexified bundle $F \otimes \mathbb{C} \rightarrow K$ to the tangent bundle TK . Then there exists a Kähler vector bundle structure on E for which the total spaces of the subbundles $F \rightarrow K$ and $iF \rightarrow K$ are real analytic, totally real and Lagrangian.*

Proof. We first describe the real Kähler structures on the tautological bundles over Grassmannians. For positive integers $m < N$ consider the action of $GL(m, \mathbb{C})$ on $\mathbb{C}^{m \times N}$ by left multiplication. We think of $\mathbb{C}^{m \times N}$ as m -tuples of (row) vectors in \mathbb{C}^N and denote by $(\mathbb{C}^{m \times N})^*$ the subset of linearly independent tuples. The maximal compact subgroup $U(m) \subset GL(m, \mathbb{C})$ acts on $\mathbb{C}^{m \times N}$ in a Hamiltonian way (for the standard symplectic structure on $\mathbb{C}^{m \times N}$) with moment map

$$\mu : \mathbb{C}^{m \times N} \rightarrow u(m), \quad M \mapsto \frac{i}{2} X X^*.$$

The quotient

$$G_{\mathbb{C}} := G_{\mathbb{C}}(m, N) = (\mathbb{C}^{m \times N})^* / GL(m, \mathbb{C}) = \mu^{-1}(i/2) / U(m)$$

is the Grassmannian of m -dimensional complex subspaces of \mathbb{C}^N ; it inherits the Kähler structure from $\mathbb{C}^{m \times N}$.

Next consider the set

$$V := \{(X, v) \in (\mathbb{C}^{m \times N})^* \times \mathbb{C}^N \mid v \in \text{span}(X)\},$$

where $\text{span}(X) \subset \mathbb{C}^N$ denotes the complex subspace spanned by the m -frame $X = (X_1, \dots, X_m)$. Since the condition $v \in \text{span}(X)$ can be expressed by complex equations – the vanishing of all $(m+1)$ -dimensional minors of the matrix (X_1, \dots, X_m, v) – V is a complex submanifold of $\mathbb{C}^{m \times N} \times \mathbb{C}^N$. The quotient

$$\gamma_{\mathbb{C}} := V/GL(m, \mathbb{C}) \rightarrow G_{\mathbb{C}},$$

where $GL(m, \mathbb{C})$ acts trivially on $v \in \mathbb{C}^N$, is the tautological rank m vector bundle over the Grassmannian $G_{\mathbb{C}}$. By construction, it inherits from $\mathbb{C}^{m \times N} \times \mathbb{C}^N$ the structure of a Kähler vector bundle.

Complex conjugation $\sigma(X, v) := (\bar{X}, \bar{v})$ defines an anti-holomorphic (i.e. $\sigma \circ i = -i \circ \sigma$) and anti-symplectic (i.e. $\sigma^* \omega_{\text{st}} = -\omega_{\text{st}}$) involution of $\mathbb{C}^{m \times N}$. Since $\sigma(UX, v) = \bar{U}\sigma(X, v)$ it descends to an anti-holomorphic and anti-symplectic involution on $\gamma_{\mathbb{C}}$. Its fixed point set, the total space of the tautological bundle

$$\gamma_{\mathbb{R}} \rightarrow G_{\mathbb{R}} := G_{\mathbb{R}}(m, N)$$

over the Grassmannian of real m -planes in \mathbb{R}^N , is therefore real analytic, totally real and Lagrangian. The map $I(X, v) := (X, iv)$ on $\mathbb{C}^{m \times N} \times \mathbb{C}^N$ is holomorphic and symplectic. Since it commutes with the action of $GL(m, \mathbb{C})$ and satisfies $I \circ \sigma = -\sigma \circ I$, it descends to a holomorphic and symplectic map on $\gamma_{\mathbb{C}}$ which anti-commutes with σ . Thus the total space of the bundle

$$i\gamma_{\mathbb{R}} := I(\gamma_{\mathbb{R}}) \rightarrow G_{\mathbb{R}}$$

(whose fibre over a real subspace $W \subset \mathbb{R}^N$ is the subspace $iW \subset \mathbb{C}^N$) is also real analytic, totally real and Lagrangian.

Now let $F \rightarrow K$ be a real vector bundle of rank m over a compact manifold K . Then for sufficiently large N there exists a continuous map $\phi : K \rightarrow G_{\mathbb{R}}$ such that $F \cong \phi^* \gamma_{\mathbb{R}}$. We equip K with a real analytic structure. Complexification yields a complex structure on the total space of the tangent bundle TK such that the zero section is real analytic. We approximate ϕ by a real analytic embedding into $G_{\mathbb{R}}$ (which is possible for N large) and complexify this to a holomorphic embedding $\phi_{\mathbb{C}} : TK \hookrightarrow G_{\mathbb{C}}$ (after replacing TK by a neighbourhood of the zero section and identifying this again with TK). By construction, ϕ is covered by an injective bundle map $\Phi : F \rightarrow \gamma_{\mathbb{R}}$. We complexify it to an injective bundle map $F \otimes \mathbb{C} \rightarrow \gamma_{\mathbb{C}}$ mapping iF to $i\gamma_{\mathbb{R}}$ and extend it to an injective bundle map $\Phi_{\mathbb{C}} : E \rightarrow \gamma_{\mathbb{C}}$ covering $\phi_{\mathbb{C}}$. Now the Kähler bundle structure on $\gamma_{\mathbb{C}}|_{\phi_{\mathbb{C}}(TK)}$ pulls back under $\Phi_{\mathbb{C}}$ to a Kähler bundle structure on $E \rightarrow TK$ with the desired properties. \square

Lemma 3.5. *Let $E \rightarrow TK$ be as in Lemma 3.4 with $\dim K = k$ and rank $E = n - k$. Then hypothesis (K) is satisfied for $X = E$, $L_0 = F$ and $L_1 = iF$.*

Proof. For the holomorphic vector bundle $\tau : E \rightarrow TK$, the projection τ and the inclusion $\iota : TK \hookrightarrow E$ of the zero section are holomorphic. Next consider $x \in K$. Pick a neighbourhood V of x in K and a real analytic trivialization $\phi : F|_V \hookrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$ mapping x to 0. Complexify it to a holomorphic embedding $\phi^{\mathbb{C}} : \mathcal{N} \hookrightarrow \mathbb{C}^k \times \mathbb{C}^{n-k}$ of a neighbourhood \mathcal{N} of $F|_V$ in E . By uniqueness of analytic continuation, the restriction of $\phi^{\mathbb{C}}$ to each $E_x \cap \mathcal{N}$ with $x \in V$ is complex linear and we can extend it linearly to the whole fibre E_x . Thus we may assume that \mathcal{N} contains $E|_V$. By construction, $\phi^{\mathbb{C}}$ maps $F|_V$ to $\mathbb{R}^k \times \mathbb{R}^{n-k}$, and by complex linearity in the fibres it maps $iF|_V$ to $\mathbb{R}^k \times i\mathbb{R}^{n-k}$. \square

Proof of Proposition 3.2. Let (X, ω) , L_0 , L_1 and K be as in the proposition. Let $F \rightarrow K$ be the normal bundle of K in L_0 and denote by $E \rightarrow TK$ the pullback bundle of $F \otimes \mathbb{C} \rightarrow K$ under the projection $TK \rightarrow K$ as in Lemma 3.4. Then a neighbourhood of K in X is diffeomorphic to a neighbourhood of K in E such that L_0 corresponds to F and L_1 to iF . Lemma 3.4 provides a Kähler vector bundle structure on E , with Kähler form ω_E , for which F and iF are Lagrangian. By Lemma 3.3, the quadruples (X, ω, L_0, L_1) and (E, ω_E, F, iF) are both isomorphic near K to the same standard model. Hence there exists a symplectomorphism from a neighbourhood \mathcal{U} of K in (X, ω) to a neighbourhood of K in (E, ω_E) mapping L_0 to F and L_1 to iF . The holomorphic structure on E pulls back to an ω -compatible integrable complex structure J on \mathcal{U} , which satisfies condition (K) by Lemma 3.5. \square

Remark. Proposition 3.2 should also hold if L is 1-sheeted near K , but the proof will be more involved in that case.

Remark. Consider a submanifold $K \subset Q$ and the immersed Lagrangian $L = Q \cup NK \subset T^*Q$ as in Example 3.1. Suppose that K admits a flat metric and has trivial normal bundle (e.g. for a 1-knot in \mathbb{R}^3). Pick a flat metric on a neighbourhood of K in Q for which K is totally geodesic and let J be the (integrable!) complex structure on a neighbourhood of K in T^*Q induced by this metric. Then local isometric coordinates for Q mapping K to \mathbb{R}^k extend to local holomorphic coordinates satisfying condition (K).

For the remainder of this section, we consider (X, L, ω, J) satisfying conditions (X), (Y), (L), (K) and (R) above.

3.3 Area and energy

Recall from [3] that the (Hofer) *energy* of a holomorphic curve f is defined as a sum of two terms,

$$E(f) := E_{\omega}(f) + E_{\lambda}(f).$$

When $f = (f_{\mathbb{R}}, f_M) : (S, \partial S, j) \rightarrow (\mathbb{R} \times M, \mathbb{R} \times \Lambda, J)$, we set

$$E_{\omega}(f) := \int_S f_M^* \omega, \quad E_{\lambda}(f) := \sup_{\varphi \in \mathcal{C}} \int_S (\varphi \circ f_{\mathbb{R}}) df_{\mathbb{R}} \wedge f_M^* \lambda,$$

where the supremum is taken over the set \mathcal{C} of nonnegative functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ with

$$\int_{\mathbb{R}} \varphi(s) ds = 1.$$

Similarly, for a holomorphic curve $f : (S, \partial S, j) \rightarrow (X, L, J)$ we define its ω -energy (or area)

$$E_{\omega}(f) := \int_{f^{-1}(\bar{X})} f^* \omega + \int_{f^{-1}(\mathbb{R}_+ \times M)} f_M^* \omega$$

and its λ -energy

$$E_{\lambda}(f) := \sup_{\varphi \in \mathcal{C}^+} \int_{f^{-1}(\mathbb{R}_+ \times M)} (\varphi \circ f_{\mathbb{R}}) df_{\mathbb{R}} \wedge f_M^* \lambda,$$

where the supremum is taken over the set \mathcal{C}^+ of all nonnegative functions $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ with

$$\int_{\mathbb{R}_+} \varphi(s) ds = 1.$$

Since the almost complex structure is comptible with ω and since J pairs the symplectization- and the Reeb direction in the ends of X , it follows that $E_{\omega}(f) \geq 0$ and $E_{\lambda}(f) \geq 0$ for any holomorphic f . Moreover, $E_{\omega}(f) = 0$ implies that either f is constant, or the image of f is contained in some cylinder over a closed Reeb orbit or in some strip over a Reeb chord.

3.4 Monotonicity and removal of singularities

Lemma 3.6 (Monotonicity Lemma). *There exist constants $\varepsilon_M, C_M > 0$ depending only on (X, L, ω, J) with the following property: For any J -holomorphic map $f : (S, \partial S) \rightarrow (X, L)$ from a (possibly noncompact) Riemann surface with boundary, passing through a point $x \in \bar{Y}$ and such that $f^{-1}(B(x, r))$ is compact for some $r < \varepsilon_M$, we have*

$$E_{\omega}(f) \geq C_M r^2.$$

Proof. The proof in Proposition 4.7.2 in [18] carries over to the present setting as follows. Since the metric is smooth, there exist constants C_0 and C_1 and $r_0 > 0$ with the following properties for $0 < r < r_0$ and any $x \in \bar{Y}$: $B(x, r) \cap L$ is contained in a contractible subset of $B(x, 2r) \cap L$, for every pair of points $x, y \in B(x, r) \cap L$ there is a curve in $B(x, 2r) \cap L$ of length at most $C_1 d(x, y)$ connecting them, and every closed curve γ in $B(x, 2r)$ bounds a disk in $B(x, 2r)$ of area at most $C_0 \ell^2(\gamma)$, where $\ell(\gamma)$ denotes the length of γ .

Assume that $x \in f(S)$. For $r < r_0$, let $S_r = f(S) \cap B(x, r)$, $\alpha_r = \partial B(x, r) \cap f(S)$, and $\beta_r = f(\partial S) \cap B(x, r)$. If r is chosen generically then α_r and β_r are collections of smooth curves. For each component α'_r of α_r we choose a curve γ'_r in $L \cap B(x, 2r)$ of length at most $C_1 l(\alpha'_r)$, where $l(\alpha'_r)$ is the length of α'_r . Then $\alpha'_r \cup \gamma'_r$ is a closed curve in $B(x, 2r)$ of length at most $(1 + C_1)l(\alpha'_r)$. Let γ_r denote the union of all curves γ'_r . Then by assumption $\alpha_r \cup \gamma_r$ bounds a collection D of disks in $B(x, r)$ of total area at most $C l^2(\alpha_r)$. Similarly, $\beta_r \cup \gamma_r$ is a cycle in the contractible

set $B(x, 2r) \cap L$, and so it bounds a surface N in $L \cap B(x, 2r)$. By Stokes' theorem

$$\int_{S_r \cup D \cup N} \omega = 0.$$

Clearly $\int_N \omega = 0$. Moreover, since ω is a calibration, $|\int_D \omega|$ is bounded by the area of D , and we conclude that

$$\int_{S_r} \omega \leq C \ell^2(\alpha_r),$$

for some constant C . Consider the distance function ρ from x . Since the norm of the gradient of ρ in the ambient manifold is 1 we conclude that $|\nabla \rho| \leq 1$ on S_r . So if we let $a(\rho)$ denote the area of S_ρ then, by Sard's theorem and the coarea formula, we have $a'(\rho) \geq \ell(\alpha_\rho)$ for almost every $\rho \leq r$. Consequently

$$\frac{d\sqrt{a}}{d\rho} = \frac{a'(\rho)}{2\sqrt{a(\rho)}} \geq \frac{1}{2\sqrt{K}},$$

for some constant K . Integrating we find $a(r) \geq C_M r^2$. \square

Remark. Alternatively, Lemma 3.6 can be proved by using Proposition 4.7.2 in [18] outside K and condition (K) near K . This reduces the lemma to the case of a holomorphic map $f = (f_1, \dots, f_n) : (S, \partial S) \rightarrow (\mathbb{C}^n, \mathbb{R}^n \cup \mathbb{R}^k \times i\mathbb{R}^{n-k})$ passing through the origin, which can be proved by considering the componentwise square $g := (f_1^2, \dots, f_n^2) : (S, \partial S) \rightarrow (\mathbb{C}^n, \mathbb{R}^n)$ with smooth Lagrangian boundary condition.

Let $D := \{z \in \mathbb{C} \mid |z| < 1\}$ and $D^+ := \{z \in D \mid \Im(z) \geq 0\}$.

Lemma 3.7 (Removal of singularities). *(a) Let $f : D \setminus \{0\} \rightarrow X$ be continuous and J -holomorphic in the interior with finite energy $E(f) < \infty$. If f is bounded, then it extends to a continuous map $D \rightarrow X$.*

(b) Let $f : (D^+ \setminus \{0\}, D^+ \cap \mathbb{R} \setminus \{0\}) \rightarrow (X, L)$ be continuous and J -holomorphic in the interior with finite energy $E(f) < \infty$. If f is bounded, then it extends to a continuous map $D^+ \rightarrow X$.

Proof. Both cases follow from the argument given right after Theorem 4.1.2 in [15], using the Monotonicity Lemma 3.6 above. \square

3.5 Asymptotics

We have the following descriptions of the asymptotic behavior of a holomorphic curve $f : (S, \partial S) \rightarrow (X, L)$ where (X, L, ω, J) satisfies conditions (X), (Y), (L), (K) and (R) above near a non-removable puncture (cf. [3, Prop 5.6]).

Proposition 3.8.

- (a) *Let $f : \mathbb{R}_+ \times S^1 \rightarrow (Y, J)$ be a holomorphic curve with $E(f) < \infty$ and suppose the image of f is unbounded. Then $f(s, t) \in \mathbb{R}_+ \times M$ for all sufficiently large s , and there exists $T > 0$ and a periodic orbit γ of the Reeb vector field of period T such that*

$$\lim_{s \rightarrow \infty} \pi_M \circ f(s, t) = \gamma(Tt), \quad \lim_{s \rightarrow \infty} \frac{\pi_{\mathbb{R}} \circ f(s, t)}{s} = T$$

in $C^\infty(S^1)$.

- (b) Let $f : \mathbb{R}_+ \times [0, 1], \mathbb{R}_+ \times \{0, 1\} \rightarrow (Y, L, J)$ be a holomorphic curve with $E(f) < \infty$ and suppose the image of f is unbounded. Then $f(s, t) \in \mathbb{R}_+ \times M$ for all sufficiently large s , and there exists $T > 0$ and a Reeb chord γ of $\Lambda \subset M$ of length T such that

$$\lim_{s \rightarrow \infty} \pi_M \circ f(s, t) = \gamma(Tt), \quad \lim_{s \rightarrow \infty} \frac{\pi_{\mathbb{R}} \circ f(s, t)}{s} = T$$

in $C^\infty([0, 1])$.

Proof. Case (a) is proved in [12]. Case (b) is proved in [1] in case $\dim(Y) = 3$ and the proof there carries over to the higher dimensional situation with only minor changes. \square

3.6 Quantization of energy

Lemma 3.9. *There exists a constant $\hbar > 0$, depending only on (X, L, ω, J) , such that for every proper J -holomorphic map $f : (S, \partial S) \rightarrow (X, L)$*

$$E_\omega(f) \geq \hbar.$$

Proof. The proof of Lemma 4.2 in [7] for symplectizations directly carries over to the relative case, giving the result for curves whose image is contained in the end $\mathbb{R}_+ \times M$. For curves whose image meets \bar{X} , the lower energy bound is guaranteed by the Monotonicity Lemma 3.6. \square

Lemma 3.10. *For every $E > 0$ there exists a constant $\hbar(E) > 0$, depending only on (X, L, ω, J) and E , such that the area of every proper J -holomorphic cylinder or strip $f : (S, \partial S) \rightarrow (X, L)$ with $E(f) \leq E$ and $E_\omega(f) > 0$ satisfies*

$$E_\omega(f) \geq \hbar(E).$$

Proof. The proof of Lemma 10.9 in [3] resp. Lemma 4.6 in [7] carries over to the relative case. \square

3.7 Holomorphic cylinders and strips of small area

Finally, we need the following generalization of a result of Hofer, Wysocki and Zehnder.

Proposition 3.11. *Given $E_0, \varepsilon > 0$ there are constants $\sigma, c > 0$ with the following properties:*

(a) *For every $R > c$ and every holomorphic cylinder $f : [-R, R] \times S^1 \rightarrow \mathbb{R} \times M$ satisfying $E_\omega(f) \leq \sigma$ and $E(f) \leq E_0$ there exists either a periodic Reeb orbit γ of period $T > 0$ such that $\pi_M \circ f(s, t) \in B_\varepsilon(\gamma(Tt))$ or some point $p \in M$ such that $\pi_M \circ f(s, t) \in B_\varepsilon(p)$ for all $s \in [-R + c, R - c]$ and all $t \in S^1$.*

(b) *For every $R > c$ and every holomorphic strip $f : ([-R, R] \times [0, 1], [-R, R] \times \{0, 1\}) \rightarrow (\mathbb{R} \times M, \mathbb{R} \times \Lambda)$ satisfying the inequalities $E_\omega(f) \leq \sigma$ and $E(f) \leq E_0$ there exists either a Reeb chord γ of length $T > 0$ such that $\pi_M \circ f(s, t) \in B_\varepsilon(\gamma(Tt))$ or some point $p \in \Lambda$ such that $\pi_M \circ f(s, t) \in B_\varepsilon(p)$ for all $s \in [-R + c, R - c]$ and all $t \in [0, 1]$.*

Proof. The proof of (a) in [13] (for the contact case) and [3] (for general stable Hamiltonian structures) carries over to case (b). \square

We will also use the following version of Lemma 5.14. of [3], whose proof carries over to the relative case using the Monotonicity Lemma 3.6.

Lemma 3.12. *Let $u_n : ([-n, n] \times [0, 1], [-n, n] \times \{1, 0\}) \rightarrow (X, L)$ be a sequence of J -holomorphic strips with*

(i) $\lim_{n \rightarrow \infty} E_\omega(u_n) = 0$, and

(ii) $\lim_{n \rightarrow \infty} u_n|_{\{\pm n\} \times [0, 1]} = p_\pm \in L$ in $C^\infty([0, 1], X)$.

Then $\lim_{n \rightarrow \infty} \text{diam } u_n([-n, n] \times [0, 1]) = 0$, and in particular $p_+ = p_-$. \square

4 Compactness

In this section, we apply the above local theory to establish a compactness result for holomorphic curves. We consider (X, L, ω, J) satisfying conditions (X), (Y), (L), (K) and (R) from the previous section. Without loss of generality we assume that the neighbourhood in condition (K) is $\mathcal{U} = \mathcal{U}_\varepsilon$, where \mathcal{U}_r denotes the open r -neighbourhood of K in X with respect to the distance induced by (ω, J) .

4.1 C^∞_{loc} -convergence

Consider a fixed connected Riemann surface (Σ, j) of finite type, by which we mean the complement of a finite set of points (the ‘‘punctures’’) in a connected compact Riemann surface with boundary. We assume that Σ is stable, meaning that its double is a stable punctured Riemann surface in the usual sense. It follows that Σ admits a unique complete hyperbolic metric h_j compatible with j such that each component of $\partial\Sigma$ is a geodesic (either closed or infinite).

Suppose $f_n : (\Sigma, \partial\Sigma, j) \rightarrow (Y, L, J)$ is a sequence of continuous maps which are J -holomorphic on $\text{int } \Sigma$ and have finite energy. Recall (Proposition 3.8) that if an interior puncture is a non-removable singularity of the map f_n , then f_n will be asymptotic to a trivial cylinder over a closed Reeb orbit in a neighborhood of that puncture. Similarly, near non-removable boundary punctures f_n is asymptotic to a trivial strip over a Reeb chord. We make the following additional assumptions on our sequence:

- (S1) Each puncture is either removable for all $n \geq 1$ or non-removable for all $n \geq 1$, and at non-removable punctures the asymptotic Reeb chords resp. closed Reeb orbits are independent of n .
- (S2) There exists a constant $C > 1$ such that for all $z \in \text{int } \Sigma$ and all $n \geq 1$ we have

$$|\nabla f_n(z)| \leq \frac{C}{\rho(z)} \quad \text{if } f_n(z) \notin \mathcal{U}_{\varepsilon/4}, \quad (2)$$

$$|\nabla(\tau \circ f_n)(z)| \leq \frac{C}{\rho(z)} \quad \text{if } f_n(z) \in \mathcal{U}_\varepsilon. \quad (3)$$

Here $\rho(z)$ denotes the injectivity radius at $z \in \Sigma$ in the hyperbolic metric, the norm is computed with respect to the hyperbolic metric on the domain and the metric determined by ω and J on the target, and $\tau : \mathcal{U}_\varepsilon \rightarrow T^*K$ is the holomorphic projection appearing in condition (K).

Proposition 4.1. *For any sequence of maps $f_n : (\Sigma, \partial\Sigma, j) \rightarrow (Y, L, J)$ satisfying (S1)-(S2) there exists a subsequence, still denoted f_n , and a map $f : (\Sigma, \partial\Sigma, j) \rightarrow (Y, L, J)$ such that*

- f is continuous on Σ and holomorphic on $\text{int } \Sigma \cup (\partial\Sigma \setminus f^{-1}(K))$,
- $f_n \rightarrow f$ in C_{loc}^0 on Σ , and
- $f_n \rightarrow f$ in C_{loc}^∞ on $\text{int } \Sigma \cup (\partial\Sigma \setminus f^{-1}(K))$.

Proof. We fix an exhaustion $B_1 \subset B_2 \subset B_3 \subset \dots$ of Σ by closed subsets $B_j := \{z \in \Sigma : d(z_0, z) \leq j\}$, where $d(z_0, z)$ denotes the distance between some fixed point z_0 and z . Since B_j is compact, the injectivity radius ρ is bounded below on it by

$$\rho_j := \min_{B_j} \rho > 0.$$

So with $C_j := C/\rho_j$ condition (S2) yields the following gradient bounds for $z \in B_j \cap \text{int } \Sigma$ and all n :

$$|\nabla f_n(z)| \leq C_j \quad \text{if } f_n(z) \notin \mathcal{U}_{\varepsilon/4}, \quad (4)$$

$$|\nabla(\tau \circ f_n)(z)| \leq C_j \quad \text{if } f_n(z) \in \mathcal{U}_\varepsilon. \quad (5)$$

We now distinguish two cases.

Case 1: The sequence $f_n(z_0)$ is unbounded.

Then, after passing to a subsequence, we have $f_n(z_0) \in \mathbb{R}_+ \times M$ with \mathbb{R} -component going to infinity. By the gradient bounds on B_j , for each fixed j we have $f_n(B_j) \subset \mathbb{R}_+ \times M$ for all sufficiently large n with \mathbb{R} -component going uniformly to infinity. Hence we can apply the usual compactness argument with smooth Lagrangian boundary conditions $\mathbb{R} \times \Lambda$ in the symplectization $\mathbb{R} \times M$.

Case 2: The sequence $f_n(z_0)$ remains in a compact subset $A \subset X$.

By the gradient bounds on B_j , for each fixed j the images $f_n(B_j)$ remain in the compact subset

$$A_j := \{x \in X \mid d(x, A \cup \bar{\mathcal{U}}_\varepsilon) \leq C_j j\}.$$

For each $z \in B_j$, we define the open ball

$$S_z := \text{int } B(z, \frac{\varepsilon}{4C_j}).$$

Now B_j is covered by a finite collection S_{z_1}, \dots, S_{z_r} of these sets.

For each of the points z_i , exactly one of the following two things happens:

- (a) after passing to a subsequence n_k , $f_{n_k}(z_i) \notin \mathcal{U}_{\varepsilon/2}$ for all $k \geq 1$, or
- (b) there exists some $N(z_i)$ such that $f_n(z_i) \in \mathcal{U}_{\varepsilon/2}$ for all $n \geq N(z_i)$.

If z_1 is of type (a), then we pass to the subsequence f_{n_k} , and if z_1 is of type (b) we pass to the subsequence f_n with $n \geq N(z_1)$. Repeating this for each index $i = 2, \dots, r$, we arrive at the situation where for each z_i either (a) or (b) holds for all $n \geq 1$.

Consider first z_i of type (a). Then the gradient bounds imply that $f_n(S_{z_i}) \cap \mathcal{U}_{\varepsilon/4} = \emptyset$ for all $n \geq 1$. So the maps $f_n : S_{z_i} \rightarrow X$ have smooth Lagrangian boundary conditions, and the usual compactness argument yields a subsequence which converges in C_{loc}^∞ (up to the boundary) to a holomorphic limit map.

Next we consider z_i of type (b). We claim that $f_n(S_{z_i}) \subset \mathcal{U}_\varepsilon$ for all $n \geq 1$. To see this, consider $z \in S_{z_i}$ and a constant speed minimal geodesic $\gamma : [0, 1] \rightarrow \Sigma$ from z_i to z , set

$$t' := \sup\{t \in [0, 1] : d(f_n(\gamma(t)), K) \leq \frac{\varepsilon}{2}\},$$

and compute

$$\begin{aligned} d(f_n(z), K) &\leq \frac{\varepsilon}{2} + d(f_n(\gamma(t')), f_n(y)) \\ &\leq \frac{\varepsilon}{2} + \int_{t'}^1 |\nabla f_n(\gamma(t))| |\dot{\gamma}(t)| dt \\ &\leq \frac{\varepsilon}{2} + d(z_i, z) \cdot \max_{t \in [t', 1]} |\nabla f_n(\gamma(t))| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4C_j} \cdot C_j \\ &\leq \varepsilon. \end{aligned}$$

This proves the claim. Now consider the holomorphic maps

$$\tau \circ f_n : S_{z_i} \rightarrow T^*K,$$

where $\tau : \mathcal{U}_\varepsilon \rightarrow T^*K$ is the holomorphic projection in condition (K). These maps have smooth Lagrangian boundary conditions on K , are uniformly bounded, and have uniform gradient bounds by condition (5) above. So the usual compactness argument yields a convergent subsequence. Denote the limit map by $g : S_{z_i} \rightarrow T^*K$.

It remains to show convergence of the components transverse to T^*K . For this, note that by condition (K) each $x \in T^*K$ has a neighbourhood U_x with a holomorphic embedding $\tau^{-1}(U_x) \hookrightarrow U_x \times \mathbb{C}^{n-k}$ sending the branches of L to $K \times \mathbb{R}^{n-k}$ and $K \times i\mathbb{R}^{n-k}$, where $k = \dim K$. Cover the image of τ by finitely many such neighbourhoods U_{x_1}, \dots, U_{x_s} and denote by $\nu_\ell : U_x \times \mathbb{C}^{n-k} \rightarrow \mathbb{C}$ the holomorphic projection onto the ℓ -th \mathbb{C} -factor, $\ell = 1, \dots, n-k$. Pulling back the U_{x_m} under g yields open subsets $S_{z_i, m} \subset S_{z_i}$ and holomorphic functions $\nu_\ell \circ f_n : S_{z_i, m} \rightarrow \mathbb{C}$ mapping the boundary $S_{z_i, m} \cap \partial\Sigma$ to $\mathbb{R} \cup i\mathbb{R}$. Since the functions are also uniformly bounded, Lemma 2.4 yields a convergent subsequence for each $\ell = 1, \dots, n-k$ and $m = 1, \dots, s$.

Combining types (a) and (b), we conclude that $f_n : B_j \rightarrow X$ has a subsequence converging in the desired sense to a continuous limit map $f_j : (B_j, B_j \cap \partial\Sigma) \rightarrow (X, L)$ which is holomorphic on $B_j \cap (\text{int } \Sigma \cup (\partial\Sigma \cap f_j^{-1}(K)))$. Finally, we take a diagonal sequence with respect to the index

j of B_j in our exhaustion to get a subsequence converging on all of Σ to a limit map $f : (\Sigma, \partial\Sigma) \rightarrow (X, L)$, with convergence in C_{loc}^0 on Σ and in C_{loc}^∞ on $\text{int } \Sigma \cup (\partial\Sigma \cap f^{-1}(K))$. \square

Remark. Note that the same proof works if we allow the domains of f_n to vary in a converging sequence of Riemann surfaces (Σ_n, j_n) .

4.2 Proof of the Compactness Theorem 1.1

For the remainder of this section, we assume familiarity with the proof of compactness for holomorphic curves in SFT presented in [3], and we will sketch how it can be adapted to our setting. We freely use the concepts and notation of [3].

We denote a *nodal Riemann surface* by (S, j, D, M) , where (S, j) is a compact Riemann surface, D is the set of double points, and M is the set of marked points. As our curves have boundary, a nodal Riemann surface will have nodes of two types: boundary nodes, where both points are on the boundary, and interior nodal points, where both are in the interior. We do not consider mixed nodes. Also, we think of the boundary components as ordered, and so the set of marked points M can be split as $M = M_{\text{int}} \cup M_1 \cup \dots \cup M_b$, where $b \geq 0$ is the number of boundary components of the surface S , and where the marked points in M_{int} are interior and the marked points in M_i lie on the i^{th} boundary component. The genus of a nodal Riemann surface with boundary is the arithmetic genus of the topological surface obtained by filling each boundary component by a disc.

We define the *signature* of a nodal Riemann surface as the sequence $\sigma = (g, b; n, m_1, \dots, m_b)$, where g is its genus, b is the number of boundary components, n is the number of interior marked points, and m_i is the number of marked points on the i^{th} boundary component.

The ε -thin part of every component of a stable nodal Riemann surface (with respect to its uniformizing metric) now consists of four types of domains: annuli of finite modulus around a short interior geodesic, annuli conformally equivalent to the punctured unit disc around each interior puncture, a rectangular region conformally equivalent to $[-1, 1] \times (-L, L)$ around each short geodesic (minimal in its free homotopy class) connecting two boundary components, and a region conformally equivalent to a punctured half-disc $D^+ \setminus \{0\}$ near each boundary puncture.

A *decoration* (i.e. an orientation reversing orthogonal identification of the tangent planes at the two corresponding points) is required only at interior nodes, since at the boundary the choice of identification is fixed by matching the boundary directions. we denote a decorated nodal Riemann surface by (S, j, D, M, r) , where r stands for the decoration.

We denote by $\overline{\mathcal{M}}_\sigma^s$ the moduli space of connected decorated stable nodal Riemann surfaces with signature σ , equipped with the usual topology ([3], cf. also [14] for the non-decorated case). It is shown in [3] that, for each fixed signature σ , the space $\overline{\mathcal{M}}_\sigma^s$ is a compact metric space which coincides with the closure of its subset \mathcal{M}_σ of smooth marked Riemann surfaces with boundary of signature σ . In other words, every sequence of smooth stable marked Riemann surfaces (S_n, j_n, M_n) of signature σ

has a subsequence which converges to a decorated nodal Riemann surface (S, j, M, D, r) of the same signature.

Now consider (X, L, ω, J) as above. With the above setup for the domains, the definition of a nodal holomorphic curve of height 1 in (X, L, J) is exactly the same as in [3, §8], except that we allow the domain to have boundary, which is required to be mapped to L . Similarly, we get the notion of a holomorphic building of height $(1|k_+)$, and the notion of convergence. Fixing the signature $\sigma := (g, b; n, m_1, \dots, m_b)$, we obtain the moduli space $\overline{\mathcal{M}}_\sigma(W, L, J)$ of stable holomorphic curves of that signature.

Now we can prove the Compactness Theorem 1.1 in the introduction, which we restate as follows.

Theorem 4.2. *Let (X, L, ω, J) satisfy conditions (X), (Y), (L), (K) and (R). Then for any $E > 0$ and for any fixed signature $\sigma = (g, b; n, m_1, \dots, m_b)$, the space $\overline{\mathcal{M}}_\sigma(X, L, J) \cap \{E(f) \leq E\}$ is compact.*

Proof. The proof closely follows the strategy of the corresponding proof of Theorem 10.2. of [3]. Clearly, it is sufficient to establish sequential compactness for smooth curves (i.e. without nodes).

So let $f_n : (S_n, \partial S_n, j_n) \rightarrow (X, L, J)$ be a sequence of curves of fixed signature and uniformly bounded energy.

Step 1: After adding additional marked points if needed, we may assume that the underlying domains $(S_n, j_n, M_n \cup Z_n)$ of the f_n are stable.

Now we want to argue that, by adding a finite set (with number depending on the energy bound) of additional pairs of points, one obtains a new sequence of stable domains, denoted by $(S_n, j_n, M_n \cup Z_n)$, such that the new sequence satisfies the gradient bounds (2) and (3).

This is based on a bubbling analysis. Indeed, to achieve (2) one argues as in [3], producing finite energy planes or spheres that each take a minimal amount $\hbar > 0$ of energy by Lemma 3.9.

So assume that (2) holds but (3) fails, i.e. there exists a sequence of points $z_n \in S_n$ such that $f_n(z_n) \in \mathcal{U}_{\varepsilon/4}$ and $\|\nabla(\tau \circ f_n)(z_n)\| \cdot \rho(z_n) \rightarrow \infty$. After passing to a subsequence, we have one of the following two cases:

- (i) $\rho'_n := \frac{\rho(z_n)}{d(z_n, \partial S_n)} \leq C < \infty$, or
- (ii) $\rho'_n \rightarrow \infty$.

In case (i), we find holomorphic embeddings $\phi_n : (D, 0) \rightarrow (S_n \setminus (M_n \cup Z_n), z_n)$ of the unit disk with

$$\frac{1}{C'} \rho'_n \leq |\nabla \phi_n| \leq C' \rho'_n$$

for some constant C' , and in case (ii) we find points $\xi_n \in D^+$ with $\xi_n \rightarrow 0$ and holomorphic embeddings $\phi_n : (D^+, D^+ \cap \mathbb{R}, \xi_n) \rightarrow (S_n \setminus (M_n \cup Z_n), L, z_n)$ of the upper half disk satisfying the same bounds.

In both cases we can modify the sequence (z_n) , rescale $f_n \circ \phi_n$ as in [3, §10.2.1] and apply Proposition 4.1 to obtain a J -holomorphic plane $f : \mathbb{C} \rightarrow X$ or half-plane $(\mathbb{H}, \mathbb{R}) \rightarrow (X, L)$ of finite energy. The map f is either proper, or it extends to a holomorphic sphere or disk by Lemma 3.7. In either case, f has area $E_\omega(f) \geq \hbar > 0$ by Lemma 3.9. Hence adding a pair of marked points and repeating this process, we obtain a bound of the form (3) after finitely many steps.

Step 2: After passing to a subsequence, the domains (S_n, M_n, Z_n) will converge to a decorated nodal Riemann surface with boundary (S, j, M, Z, D, r) . In Step 1 we have arranged for assumption (S2) to hold for our sequence, and using the energy bound we can arrange (S1) after passing to a subsequence. So, by Proposition 4.1, for a further subsequence we obtain C_{loc}^∞ -convergence on each component of the complement of the pinching geodesics in (S_n, j_n, M_n, Z_n) of the maps f_n to some limiting map f defined on the corresponding components of (S, j, M, Z, D, r) .

Step 3: Now we have to analyse the convergence in the thin part. Here, as in [3], one considers each type of component of the thin part separately. Annuli near interior marked points and near interior nodes are treated in detail in [3, §10.2.3]. In the other two cases one proceeds analogously, with the following adaptations.

Behavior near a boundary node. As in the case of interior nodes described in [3], boundary nodes appear as a result of degeneration of some component of the thin part of the S_n . The associated holomorphic strips $u_n = f_n \circ \phi_n$, obtained by precomposing with suitable uniformizations ϕ_n whose domains are longer and longer strips, have gradient bounds of the form

$$\begin{aligned} |\nabla u_n(z)| &\leq C & \text{if } u_n(z) \notin \mathcal{U}_{\varepsilon/4}, \\ |\nabla(\tau \circ u_n)(z)| &\leq C & \text{if } u_n(z) \in \mathcal{U}_{\varepsilon}. \end{aligned} \quad (6)$$

This follows by the same argument as that for equation (35) in [3]. After passing to a subsequence, the areas $E_\omega(f_n)$ converge to either zero or some positive constant.

First consider the case of zero limiting E_ω -energy. If one of the asymptotics for the limit map f in the adjacent thick parts of S is a Reeb chord, one uses part (b) of Proposition 3.11 to conclude that the other asymptotic equals the same Reeb chord. If both adjacent asymptotics are points $p_\pm \in L$ one uses Lemma 3.12 to conclude that $p_+ = p_-$.

If the limiting E_ω -energy of the strips is positive, in view of the gradient bounds (6) there can be no bubbling, and so the only possibility is breaking into a sequence of holomorphic strips. By Lemma 3.10, each nontrivial strip carries area at least $\hbar(E)$, so there can only be finitely many of them.

Behavior near a boundary puncture. Here, the adjustments are similar in nature to the ones described for the previous case, and we omit the details.

After Step 3 is done, we have a subsequence f_n of the original sequence of holomorphic curves converging to a limiting map f defined on some nodal Riemann surface (S, j, M, Z, D, r) such that $\lim E(f_n) = E(f)$.

Step 4: It remains to recover the level structure in the holomorphic building f constructed above, and this is done exactly as in [3, § 10.2.5]. \square

5 Proof of the Finiteness Theorem 1.2

As before, we consider (X, L, ω, J) satisfying conditions (X), (Y), (L), (K) and (R). Now we assume in addition that $(X, L, \omega = d\lambda)$ is *exact with convex end*, i.e. λ is a positive contact form on M which extends as a primitive of ω to \bar{X} .

For a holomorphic curve $f : (S, \partial S, j) \rightarrow (X, L, J)$ a *switch* is a point in ∂S which is mapped to K .

Now we can prove the Finiteness Theorem 1.2 in the introduction, which we restate for convenience.

Theorem 5.1. *In the situation of Theorem 1.1, suppose in addition that $(X, L, \omega = d\lambda)$ is exact with convex end. Then for each $s \in \mathbb{N}$ and $C > 0$ there exists a constant $\kappa(s, C) \in \mathbb{N}$ such that every holomorphic disk $f : (\dot{D}, \partial \dot{D}, j) \rightarrow (X, L, J)$ with at most s boundary punctures and energy $\leq C$ has at most $\kappa(s, C)$ switches.*

Proof. We argue by contradiction. So assume there exists a sequence of holomorphic disks $f : (\dot{D}, \partial \dot{D}, j) \rightarrow (X, L, J)$ with at most s boundary punctures and energy $\leq C$ such that $f_n^{-1}(K) \cap \partial D$ contains at least n points. After passing to a subsequence, we may assume that the number s of boundary punctures and the ordered collection of asymptotic Reeb chords $\Gamma = (\gamma_1, \dots, \gamma_s)$ is fixed in the sequence.

By Theorem 4.2 in the previous section, some subsequence of the f_n converges to a stable holomorphic curve f of some finite height $(1|k)$, whose domain is a disc-like nodal Riemann surface (S, j, Z, D, r) with $Z \subset \partial S$ of cardinality s . The convergence is in C^0 and in C_{loc}^∞ away from the punctures, the nodes and $f^{-1}(K) \cap \partial S$.

Consider a component C of S on which f is non-constant. We claim that in this case $f^{-1}(K) \cap \partial C$ is finite. To see this, suppose otherwise. Since f tends to infinity near the boundary punctures the set $f^{-1}(K) \cap \partial C$ avoids a neighbourhood of the punctures and thus has a limit point $p \in \partial C$. Pick a neighbourhood S_p of p which is mapped into a neighbourhood as in condition (K2) on which we have holomorphic coordinates mapping the branches of L to \mathbb{R}^n and $\mathbb{R}^k \times i\mathbb{R}^{n-k}$. Consider for $\ell = k+1, \dots, n$ the holomorphic map $\nu_\ell \circ f : S_p \rightarrow \mathbb{C}$, where $\nu_\ell : \mathbb{C}^n \rightarrow \mathbb{C}$ is the projection onto the ℓ -th \mathbb{C} -factor in these coordinates. Since $\nu_\ell \circ f$ has infinitely many zeroes in S_p , Lemma 2.5 implies that it vanishes identically, so $f(S_p) \subset K^\mathbb{C}$, where $K^\mathbb{C} \subset \mathcal{U}_\varepsilon$ is the complexification of K in condition (K). By unique continuation, the component of p in $f^{-1}(\mathcal{U}_\varepsilon)$ is mapped into $K^\mathbb{C}$, so in particular the (connected) boundary of C is mapped entirely into K . But since L is exact, the boundary of a nonconstant component must contain at least one positive puncture. This contradiction completes the proof of the claim.

It follows that $f^{-1}(K) \cap \partial S$ consists of finitely many points and finitely many components on which f takes a constant value on K . Pick disjoint compact sets S_1, \dots, S_r with piecewise smooth boundary such that $f^{-1}(K) \cap \partial S \subset \cup_i \text{int } S_i$ and for each S_i one of the following holds:

- (a) S_i contains precisely one point of $f^{-1}(K) \cap \partial S$ and no nodes, or
- (b) S_i contains precisely one connected union of components on which f takes a constant value on K .

Moreover, we may assume that each S_i is mapped into the neighbourhood \mathcal{U}_ε of K in condition (K). Note that for $n \geq N$ sufficiently large we have $f_n(\partial D \setminus \cup_i S_i) \cap K = \emptyset$ by the C^0 -convergence on compact sets. Since $f_n^{-1}(K) \cap \partial D$ contains at least n points, it follows that in some S_i the map f_n has at least n/r points of ∂D mapping to K .

Suppose first that this S_i is of type (a). Then $f_n \rightarrow f$ in C^∞ on S_i and (composing as above with projections to \mathbb{C}) Lemma 2.5 implies that f maps S_i into $K^\mathbb{C}$. As above, this yields a contradiction.

Finally, suppose that S_i is of type (b). Then we argue as in the proof of Lemma 2.5: By Lemma 2.3, the winding number of f_n over $\Gamma_i := \partial S_i \setminus (S_i \cap \partial S)$ satisfies

$$w(f_n, \Gamma_i) \geq \frac{n}{4r} \xrightarrow{n \rightarrow \infty} \infty.$$

On the other hand, the smooth convergence $f_n \rightarrow f$ on Γ_i implies

$$w(f_n, \Gamma_i) \xrightarrow{n \rightarrow \infty} w(f, \Gamma_i) < \infty.$$

This contradiction completes the proof of the Finiteness Theorem. \square

Corollary 5.2. *In the situation described above, for every ordered collection of Reeb chords $\Gamma = (\gamma_1, \dots, \gamma_s)$, there exists a constant $\kappa = \kappa(\Gamma)$ with the following property. If $Z \subset \partial D^2$ has cardinality s and $f : (D^2 \setminus Z, \partial D^2 \setminus Z) \rightarrow (T^*\mathbb{R}^3, L)$ is a J -holomorphic disc with asymptotics Γ , then $f^{-1}(K) \cap \partial D^2$ contains at most κ points. In particular, f has at most κ switches.*

A Dimensions of moduli spaces

Consider a quadruple (X, L, ω, J) satisfying conditions (X), (Y), (L), (K) and (R) in Section 3. In this appendix we give the dimension formula for moduli spaces of holomorphic curves in X with *smooth* boundary on L , interior punctures asymptotic to closed Reeb orbits, boundary punctures asymptotic to Reeb cords, and *Lagrangian intersection punctures*, i.e. boundary punctures asymptotic to the clean self-intersection K . Before proving the formula we introduce the (topological) data needed to state it.

Consider a Lagrangian intersection puncture mapping to a point k in a component K_d of K , where $\dim(K_d) = n - d$. We restate the relevant part of condition (K) as follows:

- (t0) Near k there exist local holomorphic coordinates $\mathbb{C}^{n-d} \times \mathbb{C}^d$ in which L corresponds to $\mathbb{R}^{n-d} \times (\mathbb{R}^d \cup i\mathbb{R}^d)$, and T^*K corresponds to $\mathbb{C}^{n-d} \times \{0\}$.

Assume that $f : (S, \partial S) \rightarrow (X, L)$ is a holomorphic map with a Lagrangian intersection puncture. Pick a local coordinate z in upper half plane \mathbb{H} on the source S , where the Lagrangian intersection puncture corresponds to $0 \in \mathbb{H}$, and local holomorphic coordinates $\mathbb{C}^{n-d} \times \mathbb{C}^d$ around $f(0)$ in the target as in (t0). In these coordinates f is expressed as

$$f(z) = (f_1(z), f_2(z)) \in \mathbb{C}^{n-d} \times \mathbb{C}^d,$$

where f_1 maps \mathbb{R} to \mathbb{R}^{n-d} , and f_2 maps \mathbb{R}_\pm to \mathbb{R}^d or $i\mathbb{R}^d$. It follows that f_1, f_2 have unique power series expansions of the form

$$f_1(z) = \sum_{j=0}^{\infty} a_j z^j, \quad f_2(z) = \sum_{j=0}^{\infty} c_j z^{j+w}, \quad (7)$$

where $a_j \in \mathbb{R}^{n-d}$ for all j , either $c_j \in \mathbb{R}^d$ for all j or $c_j \in i\mathbb{R}^d$ for all j , $c_0 \neq 0$, and w is either a positive half integer (if the map f switches local sheets of L at k) or a positive integer (if f remains on one sheet). We call w the *asymptotic winding number* of f at the Lagrangian intersection puncture k . (This notion is clearly independent of the choices involved in its definition).

Assume that the holomorphic map $f: (S, \partial S) \rightarrow (X, L)$ has

- p positive interior punctures at Reeb orbits $\gamma_1, \dots, \gamma_p$,
- q negative interior punctures at Reeb orbits β_1, \dots, β_q ,
- s positive boundary punctures at Reeb chords c_1, \dots, c_s ,
- t negative boundary punctures at Reeb chords b_1, \dots, b_t , and
- l Lagrangian intersection punctures on the boundary mapping to clean self intersection components K_{d_1}, \dots, K_{d_l} with asymptotic winding numbers w_1, \dots, w_l , respectively, where $\dim(K_{d_j}) = (n - d_j)$, $j = 1, \dots, l$.

We trivialize TX along parts of the map f as follows.

- (t1) Fix complex trivializations Z_γ of the contact planes in the convex end of X along all Reeb orbits $\gamma \in \{\gamma_1, \dots, \gamma_p\}$.
- (t2) Fix complex trivializations Z_β of the contact planes in the concave end of X along all Reeb orbits $\beta \in \{\beta_1, \dots, \beta_q\}$.

If $\alpha \in \{\gamma_1, \dots, \gamma_p\}$ or $\alpha \in \{\beta_1, \dots, \beta_q\}$ then the linearized Reeb flow induces a 1-parameter family of symplectomorphisms $\Phi_t: \xi_{\alpha(0)} \rightarrow \xi_{\alpha(t)}$, where $\xi_{\alpha(t)}$ is the contact hyperplane at $\alpha(t)$, $t \in [0, T]$. Using the trivialization Z_α from (t1) or (t2), we view Φ_t as a path of symplectomorphisms $\Phi_t^{Z_\alpha}: \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$. Write

$$\mu_{CZ}(\alpha, Z_\alpha) \tag{8}$$

for the Conley-Zehnder index of the path $\Phi_t^{Z_\alpha}$, $0 \leq t \leq T$ (see [8]).

Remark. [cf. [8]] The Conley-Zehnder index of a path $\Psi_t: \mathbb{C}^m \rightarrow \mathbb{C}^m$, $0 \leq t \leq 1$ is the Maslov index of the path of Lagrangian planes in $\mathbb{C}^m \oplus \mathbb{C}^m$ corresponding to the graph of Ψ_t . The Maslov index of a path L_t , $0 \leq t \leq 1$, of Lagrangian planes in \mathbb{C}^k equals $\langle \mu, [\hat{L}] \rangle - \frac{k}{2}$, where μ is the Maslov class and where \hat{L} is the loop of Lagrangian planes obtained by closing L by a positive rotation taking L_1 and L_0 . Here a positive rotation is defined as follows. Two Lagrangian subspaces V_0 and V_1 in \mathbb{C}^k defines a decomposition $W = W^1 \oplus \dots \oplus W^r$ into orthogonal subspaces and a complex angle $(\theta_1, \dots, \theta_r)$, $0 \leq \theta_1 < \theta_2 < \dots < \theta_r < \pi$ as follows. Let θ_1 be the smallest number in $[0, \pi)$ such that

$$\dim \left((e^{i\theta_1} \cdot V_0) \cap V_1 \right) \geq 1.$$

Let $W^1 \subset \mathbb{C}^k$ be the complex subspace generated by $e^{i\theta_1} \cdot V_0$ and let W' be its orthogonal complement. Then $V'_0 = W' \cap (e^{i\theta_1} \cdot V_0)$ and $V'_1 = W' \cap V_1$ are Lagrangian subspaces. Let θ'_1 be the smallest number in $(0, \pi)$ such that

$$\dim \left((e^{i\theta'_1} \cdot V'_0) \cap V'_1 \right) \geq 1.$$

Let $\theta_2 = \theta'_1 + \theta_1$ and let $W_2 \subset W' \subset W$ be the complex subspace generated by $e^{i\theta'_1} \cdot V'_0$. Repeating this construction we get a decomposition and complex angles as claimed.

The *positive rotation* taking V_0 to V_1 is the 1-parameter family of linear transformations which acts by multiplication by $e^{i\theta_j t}$, $t \in [0, 1]$ on W^j , $j = 1, \dots, r$. The *negative rotation* taking V_0 to V_1 acts by multiplication by $e^{-i(\pi - \theta_j)t}$, $t \in [0, 1]$ on W^j , $j = 1, \dots, r$.

- (t3) Fix complex trivializations Z_c of the contact planes along all Reeb chords $c \in \{c_1, \dots, c_s\}$ of the Legendrian submanifold in the convex end which have the property that the linearized Reeb flow along the chord c expressed in Z_c is constantly equal to the identity.
- (t4) Fix complex trivializations Z_b of the contact planes along all Reeb chords $b \in \{b_1, \dots, b_t\}$ of the Legendrian submanifold in the concave end which have the property that the linearized Reeb flow along the chord b expressed in Z_b is constantly equal to the identity.

Completing these trivializations with a vector field in the symplectization direction we get trivializations of TX along any Reeb orbit and along any Reeb chord appearing as asymptotic data for f .

- (t5) Fix complex trivializations Z_C of f^*TX along each component C of the complement of the punctures in ∂S with the following properties. If an endpoint of C is a Reeb chord puncture at a Reeb chord a then $Z_C = Z_a$ at the corresponding Reeb chord endpoint in some neighborhood of the endpoint of C . If a Lagrangian intersection puncture is the common endpoint of boundary components C and C' then $Z_C = Z_{C'}$ at the common endpoint.

The choices (t1) – (t2) give trivializations of f^*TX near each interior puncture in S and the choices (t3) – (t5) give a trivializations $Z_{\partial_j f}$ of f^*TX along the j^{th} component C_j of the boundary ∂S , where we think of punctures as marked points so that ∂S becomes a closed 1-manifold. Let

$$c_1^{\text{rel}}(u^*(TX); Z_{\partial f}; Z_{\gamma_1}, \dots, Z_{\gamma_p}; Z_{\beta_1}, \dots, Z_{\beta_q}) \quad (9)$$

denote the obstruction to extending this trivialization over S . Here we think of the obstruction as the number arising from evaluating the obstruction class on the orientation class of $(S_0, \partial S_0)$, where S_0 is the surface obtained from S by removing small open disks around all its interior punctures and where the bundle is trivialized along ∂S_0 .

Let Λ denote a Legendrian submanifold at one of the ends of X and let $a \in \{c_1, \dots, c_s\}$ or $a \in \{b_1, \dots, b_t\}$ be a Reeb chord of Λ . Let a^- denote the endpoint of a where the Reeb vector field points into a , and let a^+ denote the other endpoint of a . The image of the tangent space $T_a - \Lambda$ under the linearized Reeb flow along a is a Lagrangian plane $(T_a - \Lambda)' \subset \xi_{a^+}$, where ξ_y denotes the contact plane at y . Assume that a is generic in the sense that the two Lagrangian subspaces $(T_a - \Lambda)'$ and $T_{a^+} \Lambda$ of ξ_{a^+} intersect transversely (after small perturbation, all Reeb chords are generic). Let

$$R_{a^+}^{\text{neg}}(a^-, a^+): \xi_{a^+} \rightarrow \xi_{a^+}$$

denote the rotation in ξ_{a+} in the negative direction which takes $(T_a - \Lambda)'$ to $T_{a+}\Lambda$. Let

$$R_{a-}^{\text{neg}}(a^+, a^-): \xi_{a-} \rightarrow \xi_{a-}$$

be defined similarly, rotating the image $(T_{a+}\Lambda)'$ of $T_{a+}\Lambda$, under the backwards linearized Reeb flow along a , in the negative direction in ξ_{a-} to $T_{a-}\Lambda$.

Let C'_j denote the complement of the punctures in the j^{th} component $C_j \subset \partial S$. Then the tangent planes to L along $f(C'_j)$ expressed in the trivializations $Z_{\partial_j f}$ constitute a collection of paths of Lagrangian planes in \mathbb{C}^n . We close these paths to a loop as follows:

- (t3') The tangent planes of $L = \Lambda \times \mathbb{R}$ at endpoints of a Reeb chord $c \in \{c_1, \dots, c_s\}$ are connected by the product of the linearized Reeb flow along c in ξ and the identity in the symplectization direction, followed by the path

$$R_{c+}^{\text{neg}}(c^-, c^+) ((T_{c-}\Lambda)') \oplus \mathbb{R} \subset \xi_{c+} \oplus \mathbb{C}.$$

- (t4') The tangent planes of $L = \Lambda \times \mathbb{R}$ at endpoints of a Reeb chord $b \in \{b_1, \dots, b_t\}$ are connected by the backwards linearized Reeb flow along b in ξ and the identity in the symplectization direction, followed by the path

$$R_{b-}^{\text{neg}}(b^+, b^-) ((T_{b+}\Lambda)') \oplus \mathbb{R} \subset \xi_{b-} \oplus \mathbb{C}.$$

- (t0') The tangent planes at a Lagrangian intersection puncture mapping to $K_d \in \{K_{d_1}, \dots, K_{d_l}\}$ of asymptotic winding number w correspond to the planes $\mathbb{R}^{n-d} \times \mathbb{R}^d$ or $\mathbb{R}^{n-d} \times i\mathbb{R}^d$ in the coordinates (t0). Connect these planes by multiplying the tangent plane of the boundary component oriented toward the puncture with the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{-sw\pi i} \end{pmatrix}, \quad 0 \leq s \leq 1, \quad (10)$$

in the local $\mathbb{C}^{n-d} \times \mathbb{C}^d$ -coordinates of (t0).

Define

$$\mu(\partial_j f, Z_{\partial_j f}) \quad (11)$$

as the Maslov index of the loop of Lagrangian subspaces in \mathbb{C}^n which corresponds to the j^{th} boundary component of S and which is constructed by closing the paths of Lagrangian planes as described in (t3'), (t4'), and (t0').

Let $\mathcal{M}(f)$ denote the moduli space of holomorphic curves in X with boundary on L , with punctures at Reeb orbits, Reeb chords, and at Lagrangian self intersection components as described above, which have the same additional structure as f (i.e. asymptotics and Lagrangian intersection punctures, including asymptotic windings), have domain diffeomorphic to $(S, \partial S)$, and which are homotopic to f through (continuous) maps respecting the additional structure. Recall that $\dim(X) = 2n$, let g denote the genus of S , and let r denote the number of boundary components of ∂S .

Theorem A.1. *With the notation from (8), (9), and (11), the formal dimension of $\mathcal{M}(f)$ is given by*

$$\begin{aligned} \dim(\mathcal{M}(f)) &= (n-3)(2-2g-r) + (s+t+l) \\ &\quad + \sum_{j=1}^p \left(\mu_{\text{CZ}}(\gamma_j, Z_{\gamma_j}) - (n-3) \right) \\ &\quad - \sum_{j=1}^q \left(\mu_{\text{CZ}}(\beta_j, Z_{\beta_j}) + (n-3) \right) \\ &\quad + \sum_{j=1}^r \mu(\partial_j f, Z_{\partial_j f}) \\ &\quad + 2c_1^{\text{rel}}(u^*(TX); Z_{\partial f}; Z_{\gamma_1}, \dots, Z_{\gamma_p}; Z_{\beta_1}, \dots, Z_{\beta_q}). \end{aligned}$$

Remark. As mentioned in Section 1, it follows from Theorem A.1 that the contribution from a Lagrangian intersection puncture mapping to a codimension d clean intersection component with asymptotic winding number w equals $1 - wd$. Here 1 is the contribution to l and $-wd$ is the contribution to the Maslov index from the rotation in (10).

Proof. We consider first the case when there are no Lagrangian intersection punctures. The formal dimension of $\mathcal{M}(f)$ equals the Fredholm index of the linearization of the $\bar{\partial}_J$ -equation at f . The source space of this operator splits into the direct sum of an infinite dimensional functional analytic space of vector fields along f and the tangent space of the space of conformal structures of the domain S of f . We denote the restriction of the linearized $\bar{\partial}_J$ -operator to the space of vector fields by $\bar{\partial}_{\text{vf}}$.

The index of $\bar{\partial}_{\text{vf}}$ remains constant as the operator is deformed through Fredholm operators. Consider the symplectization direction in f^*TX near any boundary puncture in S . The boundary condition in this direction is degenerate. In order to describe a neighborhood of the map f in a functional analytic setting (e.g. a polyfold neighborhood of f), one would use a Sobolev space with small positive exponential weights at the punctures and augment that space by one cut-off solution corresponding to translations in the \mathbb{R} -direction for each puncture. (With notation as above, if θ_m and θ_M denotes the smallest and largest complex angles respectively of $(T_a - \Lambda)'$ and $T_a + \Lambda$ over all Reeb chords $a \in \{c_1, \dots, c_p\} \cup \{b_1, \dots, b_t\}$ then the weight being small means that it is smaller than $\min\{\theta_m, \pi - \theta_M\}$.) However, for index purposes, this is equivalent to forgetting the cut-off solution and changing the weight to a small negative exponential weight. We will work in the setting of small negative exponential weight without auxiliary solutions below.

The first deformation of $\bar{\partial}_{\text{vf}}$ will change the ξ -directions of the boundary condition near boundary punctures so that they look like the symplectization direction. Consider the boundary condition at a boundary puncture mapping to a Reeb chord a . We deform it as follows. Rotate the image of the tangent space $(T_a \pm \Lambda)'$ at the endpoint a^\mp of a boundary arc of f under the linearized Reeb flow along a (forwards or backwards according to the sign of the boundary puncture) in the negative direction to $T_{a^\mp} \Lambda$ and simultaneously change the weight at this puncture in the

ξ -directions from its initial value $1 = e^0$ to a small *negative* exponential weight. It is straightforward to check that this gives a path of Fredholm operators, compare [8, Proposition 6.14]. Denote the operator at the endpoint of this path $\bar{\partial}'_{vf}$. Consider the surface \hat{S} which is S with boundary punctures erased. Since the change of coordinates taking a neighborhood $[0, \infty) \times [0, 1]$ of a puncture in S to a neighborhood of $0 \in \mathbb{H}$ of the corresponding point in \hat{S} is $w \mapsto e^{-\pi w}$ it follows that the index of the operator $\bar{\partial}'_{vf}$ on S equals the index of the $\bar{\partial}$ -operator on the surface \hat{S} , with boundary condition naturally induced from the boundary condition of $\bar{\partial}'_{vf}$, see [8, Proposition 6.13]. We denote this operator on \hat{S} by $\bar{\partial}_{\hat{S}}$. By definition, in the trivialization $Z_{(\partial_j f)}$ along the j^{th} boundary component of \hat{S} , the Lagrangian boundary condition of $\bar{\partial}_{\hat{S}}$ has Maslov index $\mu(\partial_j f; Z_{\partial_j f})$.

We next consider interior punctures. Also here, the asymptotic operator is degenerate in the symplectization direction. As in the case of boundary punctures, one would use small positive exponential weights and cut-off solutions to define functional neighborhoods, but in order to compute the index we might as well use small negative exponential weights and no cut-off solutions to compute the index. (Here small refers to small when compared to the distance between the eigenvalues of the linearized return maps and 1.)

Fix capping spheres of all Reeb orbits at interior punctures. A capping sphere of a positive (negative) puncture where f is asymptotic to a Reeb orbit α is a once punctured sphere with a trivial $\mathbb{C}^{n-1} \oplus \mathbb{C}$ -bundle over it with trivialization which extends the trivialization Z_α given near the puncture and with a $\bar{\partial}$ -operator with the asymptotics of a negative (positive) puncture at α in the \mathbb{C}^{n-1} -direction and with trivial asymptotics and small positive exponential weight in the symplectization direction corresponding to \mathbb{C} . Thus, the capping operator $\bar{\partial}_\alpha^+$ of a positive puncture at α has index

$$\text{ind}(\bar{\partial}_\alpha^+) = (n-1) - \mu_{CZ}(\alpha, Z_\alpha),$$

and the capping operator $\bar{\partial}_\alpha^-$ of a negative puncture at α has index

$$\text{ind}(\bar{\partial}_\alpha^-) = (n-1) + \mu_{CZ}(\alpha, Z_\alpha),$$

see [17, 4]. A well-known argument shows that the index is additive under linear gluing of operators. We make one remark concerning this result in the present setup: the symplectization \mathbb{C} -component of the operator on a gluing neck limits to the standard operator on the infinite cylinder with positive exponential weight at one end and negative exponential weight at the other. This Fredholm operator is invertible and the usual linear gluing argument applies.

The result of gluing the capping spheres at the punctures of \hat{S} and the capping operators to the operator $\bar{\partial}_{\hat{S}}$ is a $\bar{\partial}$ -operator $\bar{\partial}_{\bar{S}}$ on a surface \bar{S} of genus g with r boundary components and a Lagrangian boundary condition along each boundary component. The complex bundle over \bar{S} comes equipped with a trivialization Z near its boundary. Let $\mu(\partial \bar{S}, Z)$ denote the total Maslov index of the Lagrangian boundary condition of the boundary measured with respect to the trivialization Z and let $c_1^{\text{rel}}(Z)$ denote the relative Chern class which is the obstruction to extending Z from $\partial \bar{S}$ to all of \bar{S} . Doubling \bar{S} as well as the operator $\bar{\partial}_{\bar{S}}$ over the

boundary of \bar{S} and applying the Riemann-Roch formula in combination with complex conjugation gives

$$\text{ind}(\bar{\partial}_{\bar{S}}) = n(2 - 2g - r) + \mu(\partial\bar{S}, Z) + 2c_1^{\text{rel}}(Z).$$

Additivity of the index then gives

$$\begin{aligned} \text{ind}(\bar{\partial}_{\text{vf}}) &= \text{ind}(\bar{\partial}_{\bar{S}}) \\ &= n(2 - 2g - r) \\ &\quad + \sum_{j=1}^p (\mu_{\text{CZ}}(\gamma_j, Z_{\gamma_j}) - (n - 1)) - \sum_{j=1}^q (\mu_{\text{CZ}}(\beta_j, Z_{\beta_j}) + (n - 1)) \\ &\quad + \mu(\partial\bar{S}, Z) + 2c_1^{\text{rel}}(Z) \\ &= n(2 - 2g - r) \\ &\quad + \sum_{j=1}^p (\mu_{\text{CZ}}(\gamma_j, Z_{\gamma_j}) - (n - 1)) - \sum_{j=1}^q (\mu_{\text{CZ}}(\beta_j, Z_{\beta_j}) + (n - 1)) \\ &\quad + \sum_{j=1}^r \mu(\partial_j f, Z_{\partial_j f}) \\ &\quad + 2c_1^{\text{rel}}(f^*(TX); Z_{\partial f}; Z_{\gamma_1}, \dots, Z_{\gamma_p}; Z_{\beta_1}, \dots, Z_{\beta_q}). \end{aligned}$$

In order to compute the dimension it remains only to compute the dimension $\dim(\mathcal{T})$ of the space \mathcal{T} of conformal structures on S . Doubling a surface with r boundary components in a similar way as above, studying the $\bar{\partial}$ -equation for vector fields along the surface which are tangent to the boundary along the boundary, and noting that each interior puncture adds 2 degrees of freedom and each boundary puncture adds 1 degree of freedom, we find that the dimension of the space of conformal structures on S equals

$$\dim(\mathcal{T}) = 3r + (s + t) + 2(p + q) - 6 + 6g.$$

We thus have

$$\begin{aligned} \dim(\mathcal{M}(f)) &= \text{ind}(\bar{\partial}_{\text{vf}}) + \dim(\mathcal{T}) \\ &= (n - 3)(2 - 2g - r) + (s + t) \\ &\quad + \sum_{j=1}^p (\mu_{\text{CZ}}(\gamma_j, Z_{\gamma_j}) - (n - 3)) - \sum_{j=1}^q (\mu_{\text{CZ}}(\beta_j, Z_{\beta_j}) + (n - 3)) \\ &\quad + \sum_{j=1}^r \mu(\partial_j f, Z_{\partial_j f}) \\ &\quad + 2c_1^{\text{rel}}(f^*(TX); Z_{\partial f}; Z_{\gamma_1}, \dots, Z_{\gamma_p}; Z_{\beta_1}, \dots, Z_{\beta_q}), \end{aligned}$$

finishing the proof in the case when there are no Lagrangian intersection punctures.

Consider next the case when there are Lagrangian intersection punctures. In order to define a functional analytic neighborhood of a map f with such punctures of given asymptotic winding number w mapping

to a codimension d component of the clean intersection, we puncture the boundary of S and identify a neighborhood of the puncture in the domain with $[0, \infty) \times [0, 1]$ by the change of variables $z = e^{-\pi w}$, $w = \tau + it \in [0, \infty) \times [0, 1]$. Then the Taylor expansion (7) gives

$$f_1(\tau + it) = \sum_{j=0}^{\infty} a_j e^{-\pi j(\tau + it)}, \quad f_2(\tau + it) = \sum_{j=0}^{\infty} c_j e^{-\pi(j+w)(\tau + it)}.$$

It follows that a neighborhood can be modeled on a Sobolev space with positive exponential weight $e^{(w - \frac{1}{100})\tau}$ augmented by the space of cut off solutions spanned by

$$\psi \cdot a_0, \psi \cdot a_1 e^{-\pi(\tau + it)}, \dots, \psi \cdot a_v e^{-\pi v(\tau + it)},$$

where ψ is a cut off function on $[0, \infty) \times [0, 1]$, where v is the largest integer smaller than $w - \frac{1}{100}$, and where $a_j \in \mathbb{R}^{n-d}$. (This augmentation space has dimension $(n-d)(v+1)$.)

At Lagrangian intersection punctures where the map switches local sheets of L , we observe that, as for Reeb chords above, we can close up the boundary condition at a Lagrangian intersection puncture by rotating $-\frac{\pi}{2}$ in \mathbb{C}^d , keeping the weight, and obtain a family of Fredholm operators.

Finally, we interpret the above weights in terms of the closed up boundary condition along $\partial \hat{S}$. In the source we use the change of variables $z = e^{-\pi w}$, $w = \tau + it \in S$ and $z \in \hat{S}$ as above. We conclude that an exponential weight in w -coordinates of magnitude $k'\pi$, where $k-1 < k' < k$ for an integer $k \geq 1$, corresponds to the condition that the sections of f^*TX and their first $k-1$ derivatives vanishes at 0 in the z -coordinates. Thus the dimension formula is obtained by applying the formula above to the boundary condition obtained by closing up the Lagrangian boundary conditions at each Lagrangian intersection puncture with a minimal negative rotation (i.e., if the map switches sheets at the puncture we rotate by $-\frac{\pi}{2}$ in the \mathbb{C}^d -factor complementary to T^*K and by 0 in the \mathbb{C}^{n-d} -factor corresponding to T^*K , and if the map does not switch sheets we rotate by 0 in both factors) and adding

$$1 - w'd$$

for each Lagrangian intersection puncture, where w' is the largest integer smaller than w . Here 1 comes from the increase in the dimension of the space of conformal structures \mathcal{T} . The theorem then follows by definition of the close up at Lagrangian intersection punctures, see (10). \square

References

- [1] C. Abbas, *Finite energy surfaces and the chord problem*, Duke Math. J. **96** (1999), no. 2, 241–316.
- [2] A. Abbondandolo and M. Schwarz, *Floer homology of cotangent bundles and the loop product*, arXiv:0810.1995 (2009).
- [3] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, E. Zehnder, *Compactness results in symplectic field theory*, Geom. Topol. **7** (2003), 799–888.

- [4] F. Bourgeois, K. Mohnke, *Coherent orientations in symplectic field theory*, Math. Z. **248** (2004), no. 1, 123–146.
- [5] D. Burns and R. Hind, *Symplectic geometry and the uniqueness of Grauert tubes*, Geom. Funct. Anal. **11**, no. 1, 1–10 (2001).
- [6] K. Cieliebak, T. Ekholm, J. Latschev and L. Ng, *in preparation*
- [7] K. Cieliebak and K. Mohnke, *Compactness for punctured holomorphic curves*, J. Symp. Geom. **3**, no. 4, 1–65 (2006).
- [8] T. Ekholm, J. Etnyre, M. Sullivan, *The contact homology of Legendrian submanifolds in \mathbb{R}^{2n+1}* J. Differential Geom. **71** (2005), no. 2, 177–305
- [9] Y. Eliashberg, A. Givental and H. Hofer, *Introduction to symplectic field theory*, GAFA 2000 Visions in Mathematics special volume, part II, 560–673.
- [10] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139**, no. 1, 173–199 (2000).
- [11] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. **114**, no. 3, 515–563 (1993).
- [12] H. Hofer, K. Wysocki and E. Zehnder, *Properties of pseudoholomorphic curves in symplectisations I: Asymptotics*, A.. Inst. Henri Poincaré **13**, no. 3, 337–379 (1996).
- [13] H. Hofer, K. Wysocki and E. Zehnder, *Finite energy cylinders of small area*, Ergodic Theory Dynam. Systems **22**, no. 5, 1451–1486 (2002)
- [14] C.-C. M. Liu, *Moduli of J-Holomorphic Curves with Lagrangian Boundary Conditions and Open Gromov-Witten Invariants for an S^1 -Equivariant Pair*, arXiv:math/0210257, v2 (2004)
- [15] D. McDuff and D. Salamon, *J-holomorphic Curves and Symplectic Topology*, AMS Colloquium Publications, Vol. 52, Providence (2004).
- [16] L. Ng, *Framed knot contact homology*, Duke Math. J. **141** (2008), no. 2, 365–406.
- [17] M. Schwarz, *Cohomology operations from S^1 -cobordisms in Floer homology* PhD thesis, ETH Zürich (1995)
- [18] J. C. Sikorav, *Some properties of holomorphic curves in almost complex manifolds*, in: Holomorphic curves in symplectic geometry, ed. Audin and Lafontaine, 165–189, Progr. Math. 117, Birkhäuser (1994).